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On the Complexity of Knock-knee channel routing with 3-terminal nets*

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Abstract

In this paper we consider a basic problem in the layout of VLSI-circuits, the channel-routing problem in the knock-knee mode. We show that knock-knee channel routing with 3-terminal nets is NP-complete and thereby settling a problem that was open for more than a decade. In 1987, Sarrafzadeh showed that knock-knee channel routing with 5-terminal nets is NP-complete (Sarrafzadeh, 1987). Furthermore, it is known that this problem is solvable in polynomial time if only 2-terminal nets are involved (Frank, 82), (Formann et al., 1993).

1 Introduction

The channel routing problem arises in the design process of VLSI circuits. A channel is a rectangular grid with top and bottom boundaries. Terminals are grid points located on the upper or lower boundary of the grid, and must be connected via wires. A k-terminal net is a set of k such terminals. The channel-routing problem can be described as follows: For a set of nets, find a set of edge-disjoint subgraphs of the grid connecting the terminals of each net, while minimizing the number of horizontal lines (tracks). Often, the number of terminals of the nets is restricted. The routing models mostly considered in the literature

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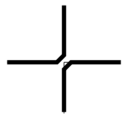

Figure 1: A knock-knee. Two nets bend at a grid vertex.

are Knock-knee routing (see, e.g. (Kuchem et al., 1996)) and Manhattan routing (see, e.g. (Middendorf, 1996)). A knock-knee is shown in Figure 1. At a knock-knee, two nets bend at a grid-vertex. Such a routing is allowed in the knock-knee model, but not in the Manhattan model. This paper is concerned with knock-knee routing. Middendorf showed that Manhattan channel-routing is NP-complete for 2-terminal nets even if all nets are single sided (*i.e.* both terminals of a net are either on the top or on the bottom boundary) and the bottom nets have density one (Middendorf, 1996). The proof of Middendorf also inspired the proof of the result in this paper. Hence Manhattan routing is harder than knock-knee routing (unless $P = NP$), since it is well known that knock-knee routing is solvable in polynomial time if only 2-terminal nets are involved (Frank, 82), (Formann et al., 1993). On the other hand, it was shown that knock-knee channel routing with 5-terminal nets is NP-complete (Sarrafzadeh, 1987), too. This paper closes the gap left open by these results by showing that knock-knee channel-routing is NP-complete for 3-terminal nets.

We start by introducing some notation and another channel routing problem in Section 2. In Section 3, it is shown that this problem is NP-complete. This result can be used directly to show the main result of the paper.

2 Preliminaries

In contrast to a channel described in Section 1, a channel with a right boundary is a rectilinear grid that has boundaries at three sides, the top boundary, the bottom boundary and the right boundary. The horizontal grid lines between top and bottom boundaries are called tracks. They are numbered $1, \ldots, k$ from the top track to the bottom track. The vertical grid lines are numbered from left to right, with the right boundary at the vertical line p.

A *terminal* is defined by a grid point on the boundary. No two terminals can be on the same grid point. In this model, terminals on the right boundary are movable in the vertical direction, *i.e.* if a terminal is specified to lie on the right boundary, the horizontal position can be chosen freely. We write t_i for a terminal that lies on the i-th vertical line and the top boundary, b_i for a terminal that lies on the i-th vertical line and the bottom boundary (for

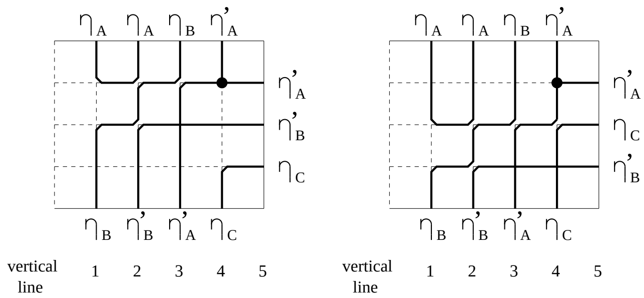

Figure 2: Two possible routings for supernets $\mathcal{N}$ as given in Example 2.1.

$i \in \{1, \ldots, p-1\}$ and r for a terminal that lies on the right boundary.

A *n-terminal net* η is a n-tuple of different terminals $\eta = (a_1, \ldots, a_n)$ where a_i is on a vertical line smaller or equal to the vertical line of a_{i+1} for $i = 1, \ldots, n-1$ (*i.e.* $a_i \in \{t_{tr_i}, b_{tr_i}, r\}$, $1 \leq tr_i \leq tr_{i+1} \leq p-1$, and $a_i \neq r$ for $i < n$). We say that a net $\eta = (a_1, \ldots, a_n)$ has its first (respectively last) terminal at vertical line l if a_1 (respectively a_n) is a terminal at vertical line l. We will often use the term "net" for a n-terminal net and omit the prefix. All nets considered in this paper are at most 3-terminal nets.

A *supernet* N is a set of nets where at most one net η in the set has a terminal at the right boundary. We say that a supernet N terminates on the right boundary if and only if there exists a net in N that has a terminal on the right boundary. A supernet N is a n-terminal supernet if all its nets are at most n-terminal nets.

Let $\mathcal{N}$ be a set of supernets for a channel with k tracks and a right boundary at vertical line p. A *routing* for $\mathcal{N}$ is an arrangement of routing paths in the channel for all the nets contained in the supernets in $\mathcal{N}$ with respect to the knock-knee model.

Example 2.1. *A channel with three tracks and a right boundary at vertical line 5 is given. Consider the set $\mathcal{N} = \{N_A, N_B, N_C\}$ of supernets where $N_A = \{\eta_A, \eta'_A\}$ with $\eta_A = \{t_1, t_2\}$ and $\eta'_A = \{b_3, t_4, r\}$, $N_B = \{\eta_B, \eta'_B\}$ with $\eta_B = \{b_1, t_3\}$ and $\eta'_B = \{b_2, r\}$, and $N_C = \{\eta_C\}$ with $\eta_C = \{b_4, r\}$. In Figure 2, two possible routings for $\mathcal{N}$ in this channel are shown.*

We can now formulate our problem.

Definition 2.1 (knock-knee channel routing with right boundary).

Instance *Given a triple $I = (k, p, \mathcal{N})$ with integers k,p and a set $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ of k 3-terminal supernets for a channel with k tracks and a right boundary at* *column p.*

Question *Is there a routing for I?*

We refer to that problem as KKRB.

The segment between two vertical lines i and $i+1$ is called *column* $i^{\rightarrow}$ or $(i+1)^{\leftarrow}$. For an instance of KKRB, the *density* of a column $i^{\rightarrow}$ (local density) is the number of nets with at least one terminal to the left (including vertical line i) and at least one terminal to the right (including vertical line $i+1$) of column $i^{\rightarrow}$. The density d of the instance (global density) is the maximum of all local densities.

For some routing R, we say that the net η is on track i in column $j^{\rightarrow}$ if track i is used by net η in column $j^{\rightarrow}$ in R. Note that a net may use several tracks in a column. We say that a supernet N is on track i in column $j^{\rightarrow}$ if some net $\eta \in N$ is routed on track i in column $j^{\rightarrow}$. Furthermore, we say that a net η changes its track at vertical line l if η is on some track i in column $l^{\leftarrow}$ and η is on some track $j \neq i$ in column $l^{\rightarrow}$. We give some easy but useful observations on knock-knee channel-routings.

Observation 2.1. *Consider a knock-knee channel routing with k tracks. Let l be a vertical line in this channel (see Figure 3). If columns $l^{\leftarrow}$ and $l^{\rightarrow}$ have density k,*

a and no net has its last terminal at l, then no net changes its track at l.

b a net η_1 has a top terminal, and a net $\eta_2 \neq \eta_1$ has a bottom terminal at l and neither η_1 nor η_2 has a last terminal at l, then η_1 is routed at a track t_1 in columns $l^{\leftarrow}$ and $l^{\rightarrow}$ and η_2 is routed at a track t_2 in columns $l^{\leftarrow}$ and $l^{\rightarrow}$ with $t_1 < t_2$.

Proof: To show Observation 2.1a, suppose that a net η is on some track t_1 in column $l^{\leftarrow}$ and on some track t_2 in column $l^{\rightarrow}$ with $t_1 \neq t_2$ (see Figure 3a). Suppose that $t_2 > t_1$. Because both columns have full density, η uses exactly one horizontal grid edge in each of these columns. Hence, η uses the vertical grid edges between t_1 and t_2. Hence, all nets that are below t_2 in column $l^{\leftarrow}$, are on tracks below t_2 in column $l^{\rightarrow}$. Denote with η' the net that is on track t_2 in column $l^{\leftarrow}$. If η has a bottom terminal at vertical line l, there is no way to route η'. If η has no bottom terminal at l, η' is routed at some track below t_2. Then there are $k - t_2 + 1$ nets routed below t_2, but only $k - t_2$ tracks below t_2. This leads to a contradiction. The proof for $t_2 < t_1$ is similar.

The situation of Observation 2.1b is shown in Figure 3b. In column $l^{\leftarrow}$, η_1 is on track t_1 and η_2 is on track t_2. Because of the full density, both nets use exactly one horizontal grid edge in $l^{\leftarrow}$ and exactly one in $l^{\rightarrow}$. Because of Observation 2.1a, these nets do not change

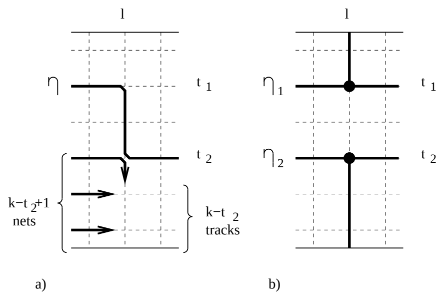

Figure 3: Three situations in a channel with full density around l. a) The situation of Observation 1. The given routing of η leads to a contradiction. b) The situation of Observation 3a. η_1 is routed above η_2 at columns $l^{\leftarrow}$ and $l^{\rightarrow}$ in any routing.

their tracks at l. Hence, the vertical grid edges from t_1 to the top boundary and from t_2 to the bottom boundary are used by η_1 and η_2, and it follow that $t_1 < t_2$. ∎

Let R be a routing for an instance $I = (k, p, \{N_1, \ldots, N_k\})$ of KKRB. For some column c, let N^c be the set of nets that are routed in column c by R *i.e.* $N^c = \{\eta | \eta \in \bigcup_{i=1}^{k} N_i$ and η is routed in column c by $R\}$. On each column c of I with full density, we define a function $tr_R^c : N^c \to \{1, \ldots, k\}$, such that for some $\eta \in N^c$, $tr_R^c(\eta)$ is the index of the track where η is routed in column c of routing R. We omit the column in the superscript if c is the last column of I (*i.e.* the column to the left of the right boundary of I). The function is defined in a similar manner for supernets of I. For some column c, let $\mathcal{N}^c$ be the set of supernets that are routed in column c by R. On each column c of I with full density, we define a function $tr_R^c : \mathcal{N}^c \to \{1, \ldots, k\}$, such that for some $N \in N^c$, $tr_R^c(N) = min\{tr_R^c(\eta) | \eta \in N$and η routed in column c by $R\}$. We take the minimum of because in our definition, a supernet could have several nets with terminals to both sides of a given column. However, we will avoid such situations threwout the paper.

For some routing R of an instance I of KKRB we say that a net η makes a *detour* in column $p^{\rightarrow}$ if eta uses at least two horizontal grid-edges in column $p^{\rightarrow}$ and either η has no terminal to the right of $p^{\rightarrow}$ (including vertica line $p + 1$) or no terminal to the left of $p^{\rightarrow}$ (including vertical line p). In the former case we also say that η makes a detour to the right at vertical line p and in the latter case we also say that η makes a detour to the left at vertical line $p + 1$. In the left figure of Figure 4 η_B makes a detour to the left at vertical line p.

Observation 2.2. *Consider an instance $I = (k, q, \mathcal{N})$ of KKRB with two nets of the following form: For $3 \le p \le q - 2$ and terminals a, b at vertical lines to the right of $p + 1$, let*

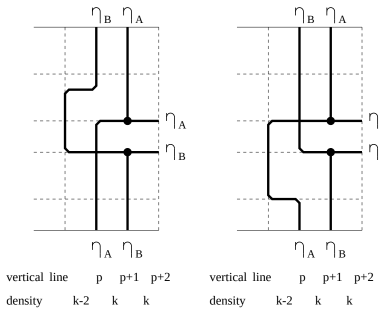

Figure 4: A channel portion described in Observation 2.2. Exactly one of the nets η_A, η_B makes a detour to the left at vertical line p. In the left figure η_B makes this detour, in the right figure η_A makes this detour.

$\eta_A = (b_p, t_{p+1}, a)$ *and* $\eta_B = (t_p, b_{p+1}, b)$. *Furthermore let the density of* $p^{\leftarrow}$ *be* $k-2$ *and the densities of* $p^{\rightarrow}$ *and* $(p+1)^{\rightarrow}$ *be* k *(see Figure 4). For any routing* R *for* I *it holds that*

a $tr_R^{(p+1)^{\rightarrow}}(\eta_B) > tr_R^{(p+1)^{\rightarrow}}(\eta_A)$,

b *at vertical line* p*, exactly one of the nets* η_A, η_B *makes a detour to the left and no other net makes a detour in column* $p^{\leftarrow}$*, and*

Proof. From Observation 2.1b it follows that $tr_R p^{\rightarrow}(\eta_A) = tr_R(p+1)^{\rightarrow}(\eta_A) < tr_R(p+1)^{\rightarrow}(\eta_B) = tr_R p^{\rightarrow}(\eta_B)$ which implies Observation 2.2a. Since the densitiy of $p^{\rightarrow}$ is k, each of these nets uses exactly one horizontal grid-edge in this column and none makes a detour to the right. Suppose that none makes a detour to the left. Then the vertical grid-edges between $tr_R p^{\rightarrow}(\eta_A)$ and $tr_R p^{\rightarrow}(\eta_B)$ are used by both nets which contradicts the definition of a layout. Hence, at least one of the nets makes a detour to the left. Since the density at column $p^{\leftarrow}$ is $k-2$ and this net uses two horizontal gird-edges at this column, k horizontal grid-edges are used and no other net in I makes a detour at column $p^{\leftarrow}$ which shows Observation 2.2b. ∎

Let $I = (k, p, \mathcal{N})$ be an instance of KKRB. An extension of I is an instance $I' = (k, q, \mathcal{N}')$ with $q > p$ and $\mathcal{N}' = \{N'_1, N'_2, \ldots, N'_k\}$ such that for all $i \in \{1, \ldots, k\}$, the following holds:

1. For all nets of the form $(a_1, \ldots, a_n)$ without a terminal on the right boundary (*i.e.* $a_n \neq r$), we have $(a_1, \ldots, a_n) \in N_i \Rightarrow (a_1, \ldots, a_n) \in N'_i$.
2. If N_i contains a net of the form $(a_1, \ldots, a_{n-1}, r)$, then N'_i contains a net of the form $(a_1, \ldots, a_{n-1}, b)$ where b is a terminal on any of the three boundaries.

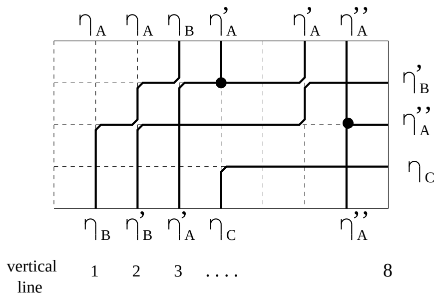

Figure 5: A routing for extension I' of Example 2.2.

To avoid cumbersome notation, we will denote the set of supernets for an instance I of KKRB and an extension I' of I with the same character. The same will be done for the corresponding supernets and the corresponding nets contained in the supernets.

Example 2.2. *Consider an instance $I = \{3, 5, \mathcal{N}\}$ of KKRB where $\mathcal{N}$ is identical to the one defined in Example 2.1. Let $I' = \{3, 8, \mathcal{N}\}$ where $\mathcal{N} = \{N_A, N_B, N_C\}$, N_B and N_C are defined as in Example 2.1, and $N_A = \{\eta_A, \eta'_A, \eta''_A\}$ with $\eta_A = \{t_1, t_2\}, \eta'_A = \{b_3, t_4, t_6\}$, and $\eta''_A = \{t_7, b_7, r\}$. I' is an extension of I. In Figure 5, a routing for I' is shown.*

With the help of extensions, we will force specific properties for the routing of some supernets, e.g. that a supernet is routed above another one in any routing. We will want that in any routing, specific nets do not change their track within the portion of the channel added by an extension. This leads to the following definition. An extension $I' = (k, q, \mathcal{N}')$ of an instance $I = (k, p, \mathcal{N})$ is $\mathcal{M}$*-safe* for I, $\mathcal{M} \subseteq \mathcal{N}$ if for every routing for I' and each $N \in \mathcal{M}$ the supernet N is on track i in column $q^{\leftarrow}$ if and only if N is on track i in column $p^{\leftarrow}$.

Consider an instance I of KKRB with supernets A and B that terminate on the right boundary. The following lemma states that there exists an extension I' that enforces B to be routed below A on the right boundary.

Lemma 2.1. *Let $I = (k, p, \mathcal{N})$ be an instance of KKRB where all k supernets in $\mathcal{N}$ terminate on the right boundary. Then, for each two different supernets $A, B \in \mathcal{N}$, there exists an extension $I' = (k, p + 4, \mathcal{N})$ of I such that:*

1. *all supernets in I' terminate on the right boundary,*

2. *there exist a routing R' for I' if and only if there exists a routing R for I with $tr_R(B) > tr_R(A)$.*

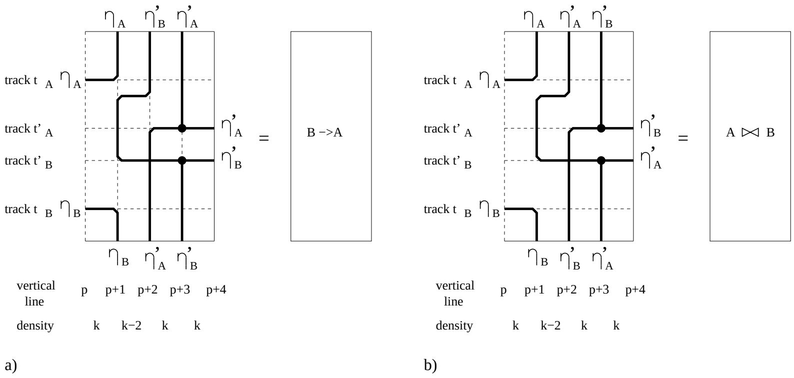

Figure 6: A layout for the supernets $A = \{\eta_A, \eta'_A\}$ and $B = \{\eta_B, \eta'_B\}$ in two extensions. Note that A is routed on a track above B at column $\leftarrow$ in any routing. a) In this extension, A is routed above B on the right boundary. b) In this extension, A is routed below B on the right boundary. Note that in (b), simply the net η'_A is exchanged with the net η'_B.

3. *If there exists a routing R' for I', then R' has the following properties:*
   (a) $tr_{R'}(B) > tr_{R'}(A)$
   (b) $tr_{R'}(B) \leq tr_{R'}^{p^{\leftarrow}}(B)$
   (c) $tr_{R'}(A) \geq tr_{R'}^{p^{\leftarrow}}(A)$.
4. *If in some routing R' for I' $tr_{R'}(A) = tr_{R'}^{p^{\leftarrow}}(A)$ and $tr_{R'}(B) = tr_{R'}^{p^{\leftarrow}}(B)$, then for each supernet $N \in \mathcal{N}$ it holds that $tr_{R'}(N) = tr_{R'}^{p^{\leftarrow}}(N)$.*

Proof. Let $\mathcal{N}$ of I' be such that the supernets in $\mathcal{N} - \{B, A\}$ are not modified (note that supernets that terminate on the right boundary of I now terminate on the right boundary of I'). The supernets B and A are modified as shown in Figure 6a: The right boundary terminal net (a_1, a_2, r) in B is replaced by a net $\eta_B = (a_1, a_2, b_{p+1})$. Furthermore we add a 3-terminal net $\eta'_B = (t_{p+2}, b_{p+3}, r)$ to B. The right boundary terminal net (a_1, a_2, r) in A is replaced by a net $\eta_A = (a_1, a_2, t_{p+1})$. Furthermore we add a 3-terminal net $\eta'_A = (b_{p+2}, t_{p+3}, r)$ to A. By construction, Condition 1 holds.

The layout of Figure 6a proves that there is a layout for I' if the Condition 2 is met. Furthermore, this routing satisfies Condition 3. The equalities can be achieved by setting $t_A = t'_A$ and $t_B = t'_B$. Note that there is no terminal at vertical line p by detinition of an instance of KKRB. Hence, no net changes its track at p.

We show that such a layout exists only if Conditions 2 and 3 are met. Clearly, there is no routing for I' if there is no routing for I. Denote the track of η_A in column $p^{\rightarrow}$ with t_A and the track of η_B in column $p^{\rightarrow}$ with t_B. Furthermore, denote the track of η'_A in column $(p+3)^{\rightarrow}$ with t'_A and the track of η'_B in column $(p+3)^{\rightarrow}$ with t'_B. The local densities of the instance are given in Figure 6a. By Observation 2.1b, $t'_A < t'_B$ follows. Hence Condition 3a holds for any routing for I'. Furthermore, $tr_{R'}^{(p+2)^{\rightarrow}}(\eta'_A) = t'_A < t'_B = tr_{R'}^{(p+2)^{\rightarrow}}(\eta'_B)$ for any routing for R' for I' by Observation 2.1b.

By Observation 2.2, a net $\eta_{left} \in \{\eta'_A, \eta'_B\}$ makes a detour to the left at vertical line $p+1$ and neither η_A nor η_B makes a detour to the right at vertical line p. Furthermore by the full density in column $p^{\rightarrow}$, η_{left} uses all vertical grid-edges between its horizontal grid-edges in $(p+1)^{\rightarrow}$ at vertical line $p+1$. It follows that in any routing it holds that $t_A \leq t'_A < t'_B \leq t_B$. This shows Conditions 2 and 3.

Consider a routing R' with $tr_{R'}(A) = tr_{R'}^{p^{\leftarrow}}(A)$ and $tr_{R'}(B) = tr_{R'}^{p^{\leftarrow}}(B)$. Proposition 4 holds for these supernets by definition. Suppose that η'_B makes the detour at verical line $p+2$. Then η'_A makes no detour, is on track t_A in column $(p+2)^{\rightarrow}$ and uses all vertical grid-edges below t_A at vertical line $p+2$. Furthermore η'_B uses all vertical grid-edges above t_A, a horizontal grid-edges at t_A and t_B in $(p+2)^{\leftarrow}$ and all vertical grid-edges between. Hence all vertical grid-edges are used at vertical lines $p+1$ and $p+2$ and no net changes its track at these vertical lines. Furthermore no net changes its track at vertical line $p+3$ by Observation 2.1a. ∎

Remark 2.3. *From Conditions 3 and 4 it follows that if $tr_R(B) = tr_R(A) + 1$ in any routing R for I, then I' is $\mathcal{N}$-safe for I. In other words, if one understands the extension merely as some part of the channel, then under these conditions, for all routings R and supernets $N \in \mathcal{N}$ it holds that $tr_R^{(p+4)^{\leftarrow}}(N) = tr_R^{p^{\leftarrow}}(N)$.*

The following lemma is very similar to Lemma 2.1. The difference in the extension is, that we enforce that A and B change their order within the extension, *i.e.* A is routed below B after the extension.

Lemma 2.2. *Let $I = (k, p, \mathcal{N})$ be an instance of KKRB where all k supernets in $\mathcal{N}$ terminate on the right boundary. Then, for each two different supernets $A, B \in \mathcal{N}$, there exists an extension $I' = (k, p+4, \mathcal{N})$ of I such that:*

1. *all supernets in I' terminate on the right boundary,*
2. *there exist a routing R' for I' if and only if there exists a routing R for I with $tr_R(B) > tr_R(A)$.*
3. *If there exists a routing R' for I', then R' has the following properties:*

(a) $tr_{R'}(B) < tr_{R'}(A)$

(b) $tr_{R'}(A) \leq tr^{p^{\leftarrow}}_{R'}(B)$

(c) $tr_{R'}(B) \geq tr^{p^{\leftarrow}}_{R'}(A)$.

4. *If in some routing R' for I', $tr_{R'}(A) = tr^{p^{\leftarrow}}_{R'}(B)$ and , $tr_{R'}(B) = tr^{p^{\leftarrow}}_{R'}(A)$ then for each supernet $N \in \mathcal{N} - \{A, B, \}$ it holds that $tr_{R'}(N) = tr^{p^{\leftarrow}}_{R'}(N)$.*

Proof. Let $\mathcal{N}$ of I' be such that the supernets in $\mathcal{N} - \{B, A\}$ are not modified. The supernets B and A are modified as shown in Figure 6b: The right boundary terminal net (a_1, a_2, r) in B is replaced by a net $\eta_B = (a_1, a_2, b_{p+1})$. Furthermore we add a 3-terminal net $\eta'_B = (b_{p+2}, t_{p+3}, r)$ to B. The right boundary terminal net (a_1, a_2, r) in A is replaced by a net $\eta_A = (a_1, a_2, t_{p+1})$. Furthermore we add a 3-terminal net $\eta'_A = (t_{p+2}, b_{p+3}, r)$ to A. By construction, Condition 1 holds. Note that with respect to the extension of Lemma 2.1, we simply exchanged the nets η'_B and η'_A. Simply by exchanging η'_B with η'_A in the proof of Lemma 2.1, Conditions 2 and 3 can be shown. This is also true for Proposition 4 as long as the nets in $\mathcal{N} - \{A, B\}$ are concerned. Note that in this case, A and B exchange their tracks within the extension which follows easily from the previous conditions on any routing R' for I'. ∎

Remark 2.4. *From Conditions 3 and 4 it follows that if $tr_R(B) = tr_R(A) + 1$ in any routing R for I, then I' is $\mathcal{N} - \{A, B\}$-safe for I.*

As a consequence of Lemma 2.1, we can enforce a particular order of supernets on the right boundary.

Lemma 2.3. *For each k, there is an instance $I = (k, p, \mathcal{N})$ of KKRB such that:*

1. *all k supernets terminate on the right boundary, and*

2. *In every routing for I, supernet N_i terminates on track i on the right boundary.*

Proof: Define an instance $I_0 = (k, k+1, \mathcal{N})$ of KKRB where $N_i = \{(t_i, r)\}$. Any order of the supernets on the right boundary can be routed in this instance. Now we extend I_0 several times using the extension of Lemma 2.1. We will use $k-1$ extension steps, where the ith step extends I_{i-1} to I_i ($i = 1, \ldots, k-1$). In the ith step, we enforce that N_{k-i+1} is below (higher track-index) N_{k-i} on the right boundary in any routing. For convenience, we divide the grid of I_{k-1} into k regions called G_0 and $G_2, \ldots, G_k$ (see Figure 7). G_0 is the grid defined by I_0, and G_i is the portion that was added by the $(k+1-i)$-th extension[1]. We show that in I_{k-1} (the last extension in our construction), N_i is at track i in the first

[1] G_i spans the columns $(6k+1-5i)^{\rightarrow}$ to $(6k-5(i-1))^{\rightarrow}$ for $i = 2, \ldots, k$.

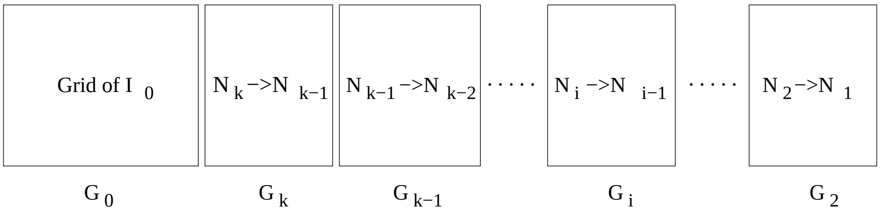

Figure 7: An instance I_{k-1} of KKRB where N_i terminates on track i in any routing. This is done with the extensions of Lemma 2.1. We divide the grid of I_{k-1} into k regions.

column of G_i.

We show by induction on i that the supernet N_i is on a track with index i or higher in the first column of G_i. This serves as the induction hypothesis. Clearly, N_1 cannot be on a track with a lower index than 1 which proves the induction basis. Suppose that for some $i \in \{2, \ldots, k\}$, N_i is on some track $t_0 < i$ in the first column of G_i (and therefore in the last column of G_{i-1}). It follows by Lemma 2.1 that in the last column of G_i, N_{i-1} is on a track $t_1 < t_0 \leq i-1$. Hence, N_{i-1} is on a track $t_1 < i-1$ in the first column of G_{i-1}, which is a contradiction.

We show by induction on the extensions and by the properties given in Lemma 2.1 that for $i = 2, \ldots, k$, in any routing, N_i is on a track with an index lower than or equal to i in the first column of G_i. Clearly, N_k cannot be on a track with index larger than k in the first column of G_k which is our induction basis. Note that the basis is at k and we conclude from i to $i+1$ in the induction step. Suppose the N_i is on a track $t_0 > i$ in the first column of G_i. In the last column of G_{i+1}, N_i is on track t_0. It follows from Lemma 2.1 that N_{i+1} is on a track $t_1 > t_0 \geq i+1$ in the first column of G_{i+1} which is a contradiction.

Now, consider a region G_i with $i \in \{2, \ldots, k-1\}$. In the first column of this region, N_i is routed at track i and in the first column of G_{i+1}, N_{i+1} is routed at track $i+1$. By Lemma 2.1, G_{i+1} is also routed at track $i+1$ in the first column of G_i. By Remark 2.3, the track of N in the first column of G_i is the same as the track of N in the last column of G_i for all nets $N \in \mathcal{N}$. It follows that N_i is on track i on the right boundary of I. ∎

We will need another type of extension. In Lemma 2.3, we used the extension of Lemma 2.1 to enforce a particular order on the right boundary of the channel. But Lemma 2.1 cannot be used to enforce a supernet to be above another one without an influence on the other supernets in the channel. We use a trick to get a similar result. For each net N_i, introduce another net $\bar{N}_i$, which we will call the *shadownet* of N_i. Typically, $\bar{N}_i$ will be on

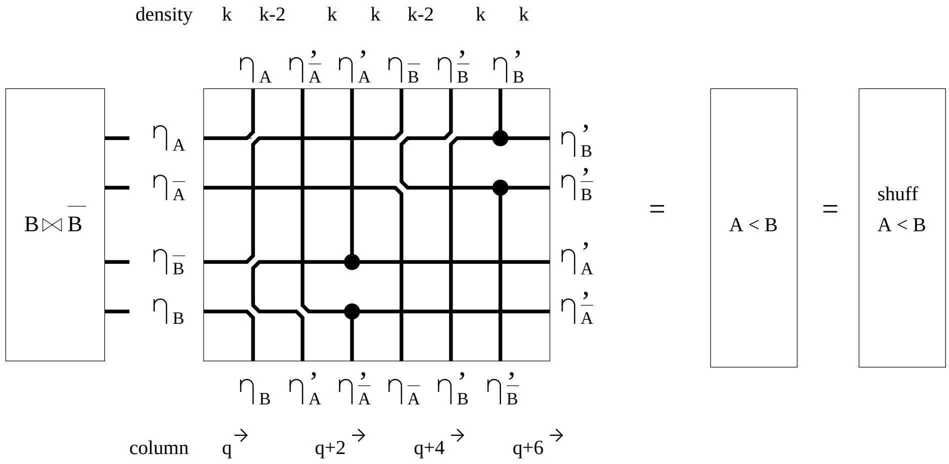

Figure 8: The shuffle-check extension and a possible routing.

a neighboring track of N_i. We say that a supernet $\bar{N}$ is a shadownet of N in an instance I of KKRB, if in every routing R for I, it holds that $tr_R(\bar{N}) = tr_R(N) + 1$.

We say that an instance I of KKRB is (A, B)-*mutable* for two supernets A, B, if for any routing R for I there is a routing R' for I, such that A and B change their tracks with respect to R on the right boundary. More formally, an instance $I = (k, p, \mathcal{N})$ of KKRB is (A, B)-*mutable* for two supernets $A, B \in \mathcal{N}$, if the following holds:

- If R is a routing for I, then there exists a routing R' for I such that $tr_{R'}(A) = tr_R(B)$ and $tr_{R'}(B) = tr_R(A)$.

Lemma 2.4. *Let $I = (k, p, \mathcal{N})$ be an instance of KKRB where all k supernets in $\mathcal{N}$ terminate on the right boundary. Then, for each four different nets $A, \bar{A}, B, \bar{B} \in \mathcal{N}$ where $\bar{A}$ is a shadownet of A and $\bar{B}$ is a shadownet of B in I, there exists an extension $I' = (k, p + 11, \mathcal{N})$ of I such that:*

1. *all supernets in I' terminate on the right boundary,*
2. *a routing exists for I' if and only if a routing R exists for I with $tr_R(A) < tr_R(B)$,*
3. *I' is (A, B)-mutable,*
4. *I' is $\mathcal{N} - \{A, \bar{A}, B, \bar{B}\}$-safe for I, and*
5. *$\bar{A}$ is a shadownet of A in I' and $\bar{B}$ is a shadownet of B in I'.*

Proof. The supernets A, $\bar{A}$,B and $\bar{B}$ are extended as shown in Figure 8. The nets η_A and

η'_A belong to supernet A, the nets η_B and η'_B belong to supernet B, the nets $\eta_{\bar{A}}$ and $\eta'_{\bar{A}}$ belong to supernet $\bar{A}$, and the nets $\eta_{\bar{B}}$, $\eta'_{\bar{B}}$ belong to supernet $\bar{B}$. Other supernets are not altered. Condition 1 is true by construction. First we extend I by exchanging the tracks of B and $\bar{B}$. Since these nets are shadownets, this step is $\mathcal{N} - \{B, \bar{B}\}$-safe for I and the right boundary is at vertical line $q = p + 4$ after this extension. Clearly, the relative order of A and B does not change by this extension.

In order to simplify the notation, we will introduce the following abbrevations: For a routing R' of I' and a supernet $N \in \{A, B, \bar{A}, \bar{B}\}$ we write t_N for $tr_{R'}^{q^{\rightarrow}}(N)$ and t'_N for $tr_{R'}(N)$. Note that in this notational convention, t_N corresponds to the track of η_N at column $q^{\rightarrow}$ and t'_N corresponds to the track of η'_N at column $(q+6)^{\rightarrow}$. First, consider the columns $(q+3)^{\rightarrow}, \ldots, (q+6)^{\rightarrow}$. This part of the channel is very similar to the extension of Lemma 2.1. We can show that

- $t'_{\bar{B}} > t'_B$,
- there is no routing R' such that $tr_{R'}^{(q+3)^{\rightarrow}}(\bar{B}) > tr_{R'}^{(q+3)^{\rightarrow}}(\bar{A})$, and
- if in any routing R', $tr_{R'}^{(q+3)^{\rightarrow}}(\bar{A}) = tr_{R'}^{(q+3)^{\rightarrow}}(\bar{B}) + 1$, then no net except $\eta_{\bar{A}}$, η'_B and $\eta'_{\bar{B}}$ changes its track in these columns.

This can be easily shown by replacing the names of the respective nets in the proof of Lemma 2.1.

Now consider the columns $q^{\rightarrow}, \ldots, (q+3)^{\rightarrow}$. This is also a similar channel-portion to the one in Lemma 2.1. So, $t_B > t_A$ in any routing for I'. There exists a routing under this condition, as shown in Figure 8. This shows Condition 2. No net can change its track at vertical line $q+3$ by Observation 2.1a. Hence, in any routing $\eta_{\bar{B}}$ is on a track above $\eta_{\bar{A}}$ at column $(q+2)^{\rightarrow}$. Furthermore, by Observation 2.2, one of the nets of $\eta'_{\bar{A}}$ and η'_A makes a detour to the left at vertical line $q+2$, and this detour uses exactly two horizontal grid-edges at column $(q+1)^{\rightarrow}$. We denote the upper horizontal grid-edge with t_u and the lower one with t_l. Because of the full density at column $q^{\rightarrow}$, this detour uses the vertical grid-edges between t_u and t_l. By Observation 2.2b, neither η_A nor η_B makes a detour at vertcal line $q+1$. Hence, $t_A \leq t_u < t_l \leq t_B$ holds for any routing.

Claim 2.1. *In any routing for I', it holds that $t_u \in \{t_A, t_{\bar{B}}\}$ and $t_l \in \{t_B, t_{\bar{A}}\}$.*

Proof (Claim 2.1): Suppose that $t_{\bar{A}} \leq t_u < t_{\bar{B}}$. Since the vertical grid-edges between t_u and t_l are used by the detour, $\eta_{\bar{A}}$ is on a track above t_u and $\eta_{\bar{B}}$ is on a track below $t_l$at column $(q+1)^{\rightarrow}$. Furthermore, $\eta'_{\bar{A}}$ uses the vertical grid-edges above t_u at vertical line

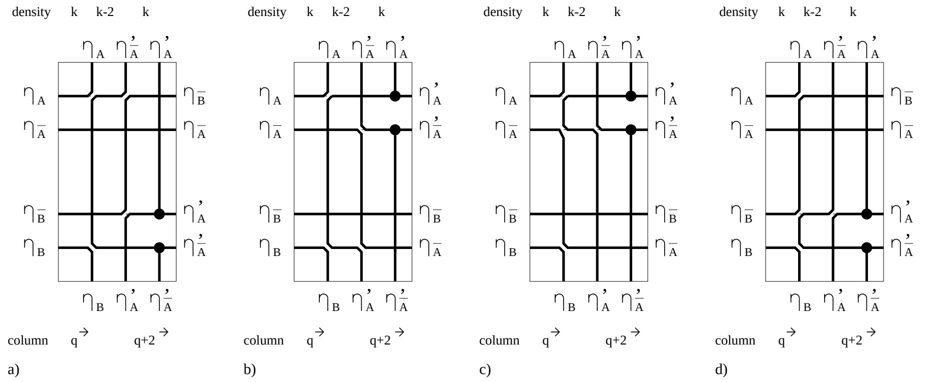

Figure 9: Four possible routings in columns $q^{\rightarrow}, \ldots, (q+2)^{\rightarrow}$ of the shuffle-check extension. We distinct three cases depending on the tracks that the detour uses at column $(q+1)^{\rightarrow}$. a) Tracks t_A and t_B are used and η'_{barA} makes the detour. b) The same tracks as in (a) are used and η'_A makes the detour. c) Tracks t_A and t_{barA} are used. d) Tracks t_B and t_{barB} are used.

$q+2$ (no more detour is possible for $\eta'_{\bar{A}}$). Hence, $\eta_{\bar{A}}$ is above $\eta_{\bar{B}}$ in column $(q+2)^{\rightarrow}$ which leads to a contradiction since there is no routing for this case. Hence, only the tracks t_A and $t_{\bar{B}}$ reamain for t_u. A similar argument shows that $t_l \in \{t_B, t_{\bar{A}}\}$. ∎

By Claim 2.1, there are three cases to consider (by definition $t_u < t_l$):

Case 1: $t_u = t_A$ and $t_l = t_B$. Suppose that η'_A makes the detour at vertical line $q+2$ (see Figure 9a). No net of $\eta_{\bar{A}}$ and $\eta_{\bar{B}}$ changes its track at vertical line $q+1$. Since at vertical line $q+2$, $\eta_{\bar{A}}$ cannot change to a track below $\eta_{\bar{B}}$, $\eta_{\bar{B}}$ changes to track t_A in order to be above $\eta_{\bar{A}}$ at column $(q+2)^{\rightarrow}$. Since η'_A is routed above $\eta'_{\bar{A}}$ at columns $(q+2)^{\rightarrow}$, $(q+3)^{\rightarrow}$, and the only possible horizontal grid-edge in this column above $\eta'_{\bar{A}}$ is at track $t_{\bar{B}}$, η'_A is routed on track $t_{\bar{B}}$ at columns $(q+2)^{\rightarrow}$, $(q+3)^{\rightarrow}$. Hence, at vertical line $q+3$, η'_A is on track $t_{\bar{B}}$ and $\eta'_{\bar{A}}$ is on track t_B. Furthermore, at this column, $\eta_{\bar{B}}$ is on track t_A and $\eta_{\bar{A}}$ is on track $t_{\bar{A}}$. It follows that in this case, $t'_B = t_A$, $t'_{\bar{B}} = t_{\bar{A}}$, $t'_A = t_{\bar{B}}$ and $t'_{\bar{A}} = t_B$ and the extension is safe for all nets without a terminal in the extended area.

Suppose that $\eta'_{\bar{A}}$ makes the detour at vertical line $q+2$ (see Figure 9b). By similar arguments, one can prove the tracks of the nets at column $(q+3)^{\rightarrow}$ as shown in Figure 9b. It follows that in this case, $t'_B = t_{\bar{B}}$, $t'_{\bar{B}} = t_B$, $t'_A = t_A$ and $t'_{\bar{A}} = t_{\bar{A}}$ and the extension is safe for all nets without a terminal in the extended area.

Case 2: $t_u = t_A$ and $t_l = t_{\bar{B}}$. η'_A uses all vertical grid-edges below t'_A and hence no net can change its track at vertical line $q+2$ (see Figure 9c). $\eta_{\bar{B}}$ cannot change to a track above $\eta_{\bar{A}}$ at vertical line $q+1$. Hence $\eta_{\bar{A}}$ changes to track t_B at $q+1$. Hence, at vertical line $q+3$, η'_A is on track $t_{\bar{B}}$ and $\eta'_{\bar{A}}$ is on track t_B. Furthermore, at this column, $\eta_{\bar{B}}$ is on track t_A and $\eta_{\bar{A}}$ is on track $t_{\bar{A}}$. It follows that in this case, $t'_B = t_A$, $t'_{\bar{B}} = t_{\bar{A}}$, $t'_A = t_{\bar{B}}$ and $t'_{\bar{A}} = t_B$ and the extension is safe for all nets without a terminal in the extended area.

Case 3: $t_u = t_{\bar{A}}$ and $t_l = t_B$. $\eta'_{\bar{A}}$ uses all vertical grid-edges above t'_B and hence no net can change its track at vertical line $q+2$ (see Figure 9d). $\eta_{\bar{A}}$ cannot change to a track below $\eta_{\bar{B}}$ at vertical line $q+1$. Hence $\eta_{\bar{B}}$ changes to track t_A at $q+1$. Hence, the supernets are on tracks given at Figure 9d. It follows that in this case, $t'_B = t_{\bar{B}}$, $t'_{\bar{B}} = t_B$, $t'_A = t_A$ and $t'_{\bar{A}} = t_{\bar{A}}$ and the extension is safe for all nets without a terminal in the extended area.

∎

We call this extension a shuffle-check extension. We use this extension in two different ways. We can use Condition 2 of the lemma to enforce that A is routed above B in column $p^{\leftarrow}$ of I'. In this case we need to take care of Conidtion 3 since as a side-effect of the extension, supernets A and B together with their shadownets may change their tracks. In this case we denote a shuffle-check extension for supernets A and B with $A < B$.

On the other hand, we can use Condition 3 to make two routings for A and B possible. Namely that A and B stay on their track or that A exchanges its track with B. In this case we need to take care that there is a routing such that A is routed above B in column $p^{\leftarrow}$. In this case we indicate the aim of the extension by labeling the symbol for the check-shuffle extension with “shuff” as shown in Figure 8.

Note that in any of the extensions introduced, nets with a right boundary are at most 3-terminal nets. Furthermore, if we introduce a last terminal in a net in an extension, no other terminal is introduced for that net in this extension. Hence, by starting with a net of the type described in Lemma 2.3 and extending this net with one of the extensions introduced in this Section, at most 3-terminal nets are used in the resulting instance.

3 The Main Theorem

We show that KKRB is NP-complete by reducing a known NP-complete problem to it. This result will be used to prove the complexity of knock-knee channel routing with 3-terminal nets.

Theorem 3.1. *KKRB is NP-complete.*

Proof: A nondeterministic algorithm can guess a routing and check if it is valid. If yes, output "yes", otherwise output "no". This can be done in polynomial time with the number of tracks and columns. So, the problem is in NP. To prove the completeness for NP, we reduce the *exactly-one-in-three* 3SAT problem to KKRB. Let a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses, each of size 3 over a set $\Sigma = \{v_1, v_2, \ldots, v_n\}$ of variables, be an instance of exactly-one-in-three 3SAT. Without loss of generality, we can assume that no clause contains a negated literal (this restriction is known to be NP-complete (Garey et al., 1979)). The exactly-one-in-three 3SAT problem asks whether there is a truth assignment of the variables in Σ such that each clause in $\mathcal{C}$ contains exactly one true literal.

The idea of the proof is as follows. We begin by constructing an instance of KKRB. We divide the tracks of the channel into five consecutive groups $G_1, \ldots, G_5$. Tracks in G_i are above tracks in G_{i+1} for $i \in \{1, \ldots, 4\}$. For each clause $C_l = \{v_h, v_i, v_j\}$, we introduce three supernets V_h^l, V_i^l, and V_j^l that terminate on tracks in group G_2 on the right boundary in every routing. Furthermore, for each variable v_i we introduce two supernets H_i and L_i that terminate on tracks in group G_3 on the right boundary in every routing. Then we extend our instance and enforce that for each variable v_i, either H_i changes to a track in group G_1 or L_i changes to a track in G_5. This will give us a truth assignment for the variables. Furthermore, we force all supernets of the form V_i^l to change to a track of group G_4 if and only if for the corresponding variable v_i the supernet L_i is on a track in group G_5. In addition we require that exactly one of the three supernets V_h^l, V_i^l, and V_j^l for a clause C_l will change to a track in group G_4. Thus, there will be a routing if and only if there exists a truth assignment for the variables satisfying $\mathcal{C}$.

We formalize these ideas. Let $k = 8n + 10m$ be the number of tracks. Recall that n is the number of variables and m is the number of clauses in the reduced problem. We divide the channel into the five groups $G_1 = \{\text{track } i | i \in \{1, \ldots, 2n\}\}$, $G_2 = \{\text{track } i | i \in \{2n + 1, \ldots, 2n + 6m\}\}$, $G_3 = \{\text{track } i | i \in \{2n + 6m + 1, \ldots, 6n + 6m\}\}$, $G_4 = \{\text{track } i | i \in \{6n+6m+1, \ldots, 6n+10m\}\}$, and $G_5 = \{\text{track } i | i \in \{6n+10m+1, \ldots, 8n+10m\}\}$. We construct an instance $I_0 = (k, p_0, \mathcal{N})$ of KKRB that sorts the supernets based on Lemma 2.3 as follows:

1. For each variable v_i, $i \in \{1, \ldots, n\}$, there is a set $\mathcal{V}_i$ consisting of 8 supernets $\mathcal{V}_i = \{A_i, \bar{A}_i, B_i, \bar{B}_i, L_i, \bar{L}_i, H_i, \bar{H}_i\}$
2. For each clause $\mathcal{C}_l = \{v_h, v_i, v_j\}$, $l \in \{1, \ldots, m\}$, there is a set $\mathcal{C}_l$ of 10 supernets $\mathcal{C}_l = \{V_h^l, \bar{V}_h^l, V_i^l, \bar{V}_i^l, V_j^l, \bar{V}_j^l, X_l, \bar{X}_l, Y_l, \bar{Y}_l\}$

Now we set $\mathcal{N} = \bigcup_{i \in \{1,\ldots,n\}} \mathcal{V}_i \cup \bigcup_{l \in \{1,\ldots,m\}} \mathcal{C}_l$. $I_0 = (k, p_0, \mathcal{N})$ is constructed such that there exists a routing for I_0 if and only if the supernets in $\mathcal{N}$ terminate with a net on the right boundary in the following ways:

1. For each variable v_i, $i \in \{1, \ldots, n\}$, the terminals on the right boundary for the supernets are as follows:
   (a) A_i, $\bar{A}_i$ are in this order on neighboring tracks in G_1 (more precisely, they are on tracks $2i - 1$ and $2i$).
   (b) L_i, $\bar{L}_i$, H_i, $\bar{H}_i$ are in this order on neighboring tracks in G_3 (more precisely, they are on tracks $2n + 6m + 4i - 3$, $2n + 6m + 4i - 2$, $2n + 6m + 4i - 1$ and $2n + 6m + 4i$).
   (c) B_i, $\bar{B}_i$ are in this order on neighboring tracks in G_5 (more precisely, they are on tracks $6n + 10m + 2i - 1$ and $6n + 10m + 2i$).
2. For each Clause $\mathcal{C}_l = \{v_h, v_i, v_j\}$, $l \in \{1, \ldots, m\}$, $h < i < j$, the terminals on the right boundary for the supernets are as follows:
   (a) $V_h^l, \bar{V}_h^l, V_i^l, \bar{V}_i^l, V_j^l, \bar{V}_j^l$ are in this order on neighboring tracks in G_2 (more precisely, they are on tracks $2n+6l-5$, $2n+6l-4$, $2n+6l-3$, $2n+6l-2$, $2n + 6l - 1$ and $2n + 6l$)
   (b) X_l, $\bar{X}_l$, Y_l, $\bar{Y}_l$ are in this order on neighboring tracks in G_4 (more precisely, they are on tracks $6n + 6m + 4l - 3$, $6n + 6m + 4l - 2$, $6n + 6m + 4l - 1$ and $6n + 6m + 4l$).

We extend our instance I_0 step by step, in such a manner that we can fix a truth assignment for the variables in Σ. One extension step is performed for each variable. Let $I_i = (k, p_i, \mathcal{N})$ be the extended instance after the ith extension. The effect of the ith extension is that there is a routing for the extended instance I_i if and only if either supernet L_i and H_i terminate on tracks in G_1 and G_3 on the right boundary, or L_i and H_i terminate on a tracks in G_3 and G_5 on the right boundary. In the first case, we assign $false$ to variable v_i. In the latter case, we assign $true$ to variable v_i. The extension I_i consists of four sub-extensions $I_{i,1}$ to $I_{i,4}$ and is shown in Figure 10 (shadownets are not shown). For notational simplicity, we denote the portion of the channel added by extension $I_{i,j}$ with $D_{i,j}$ ($i \in \{1, \ldots, n\}, j \in \{1, \ldots, 4\}$), as shown in Figure 10.

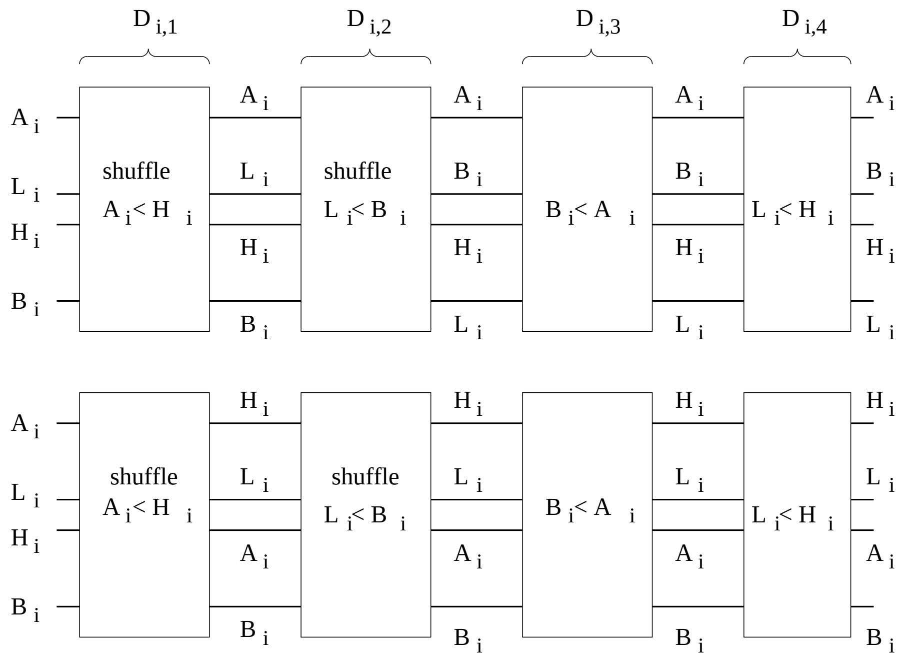

Figure 10: An extension that is used to produce a truth assignment for variable v_i and two possible routings (shadownets are not shown).

Claim 3.1. *For all $i \in \{1, \dots, n\}$ it holds that:*

1. *Extension I_i is $(\mathcal{N} - \mathcal{V}_i)$-safe for I_{i-1}.*

2. *All supernets of I_i terminate on the right boundary of I_i.*

3. *I_i is (L_i, H_i)-mutable and $\bar{L}_i, \bar{H}_i$ are shadownets of L_i and H_i respectively.*

4. *In every routing for I_i, either*

   (a) *exactly one supernet of $\{L_i, H_i\}$ terminates on a track in G_1 and exactly one of them terminates on a track in G_3 on the right boundary, or*

   (b) *exactly one supernet of $\{L_i, H_i\}$ terminates on a track in G_3 and exactly one of them terminates on a track in G_5 on the right boundary.*

5. *For all cases in Condition 4, there exists such a routing.*

Proof(Claim 3.1): Since any sub-extension $I_{i,j}$ of I_i is $\mathcal{N} - \mathcal{V}_i$-safe for $I_{i,j-1}$ respectively I_{i-1}, I_i is $\mathcal{N} - \mathcal{V}_i$-safe for I_{i-1}. By construction, all supernets of I_i terminate with a 2- or

3-terminal net on the right boundary of I_i. By construction of $D_{i,4}$, I_i is (L_i, H_i)-mutable and $\bar{L}_i$, $\bar{H}_i$ are shadownets of L_i and H_i respectively.

By construction of I_{i-1}, the arrangements of the supernets in the last column of I_{i-1} is as given in Figure 10. A_i is on a track in G_1, L_i is on a track above H_i in G_3, and B_i is on a track in G_5.

By Lemma 2.4, $I_{i,1}$ is $\mathcal{N} - \{A_i, \bar{A}_i, H_i, \bar{H}_i\}$-safe for I_{i-1}, and we have to consider two types of routings: whether A_i changes its track with H_i within $D_{i,1}$. Suppose that A_i does not change its track with H_i within $D_{i,1}$ (upper figure in Figure 10). Then, there exists no routing such that L_i stays on its track within $D_{i,2}$ (otherwise L_i would be above H_i in the first column of $G_{i,4}$ and hence there is no routing for I_i by Lemma 2.4). Hence L_i change its track with B_i within $D_{i,3}$ and in this case, L_i and H_i are on tracks in $G_3 \cup G_5$ in the first column of $D_{i,4}$ and H_i is on a track above L_i. By definition of extension $I_{i,4}$, this holds for the right boundary of I_i.

Suppose that A_i changes its track with H_i within $D_{i,1}$ (lower figure in Figure 10). Then, if B_i changes its track with L_i within $D_{i,2}$, B_i is above A_i in the first column of $D_{i,4}$. It follows by Lemma 2.4, that there is no such routing for I_i. Hence, A_i does not change its track with L_i in the second extension, and L_i, H_i are on tracks of $G_3 \cup G_5$ on the right boundary of I_i. ∎

Now, we extend our instance I_n step by step in such a way that we can choose exactly one true variable in each clause. One extension step is performed for each clause. Let I_{n+l} be the instance after the lth extension step. The effect of the lth extension step will be that there is a routing for I_{n+l} if and only if exactly one of the three supernets V_h^l, V_i^l, V_j^l changes to a track in G_4. The extension I_{n+l} is given in Figure 11 (shadownets are not shown). I_{n+l} consists of four sub-extensions $I_{n+l,1}, \ldots, I_{n+l,4}$, and we denote the portion of the cannel added by extension $I_{n+l,j}$ with $D_{n+l,j}$ ($l \in \{1, \ldots, m\}, j \in \{1, \ldots, 4\}$).

Claim 3.2. *For each clause $C_l = \{v_h, v_i, v_j\} \in \mathcal{C}$, the following holds:*

1. *Extension I_{n+l} is $\mathcal{N} - \mathcal{C}_i$-safe for I_{n+l-1}.*

2. *All supernets of I_{n+l} terminate on the right boundary of I_i.*

3. *In each routing for I_{n+l} exactly one of the three supernets V_h^l, V_i^l, or V_j^l terminates on a track in G_4, and the other two terminate on a track in G_2 on the right boundary. Furthermore in I_{n+l}, $\bar{V}_h^l, \bar{V}_i^l$, and $\bar{V}_j^l$ are shadownets of V_h^l, V_i^l and V_j^l respectively .*

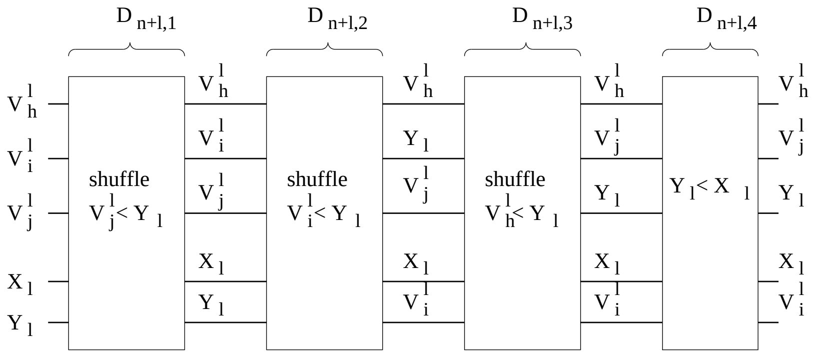

Figure 11: An extension that forces that exactly one supernet out of V_h^l, V_i^l, and V_j^l to a track in G_4. The interpretation is, that this variable is true.

4. *For all cases in Condition 4, there exists such a routing.*

Proof(Claim 3.2): Since any sub-extension $I_{n+l,j}$ of I_{n+l} is $\{\mathcal{N} - \mathcal{C}_l\}$-safe for $I_{n+l,j-1}$ respectively I_{n+l-1}, I_{n+l} is $\mathcal{N} - \mathcal{V}_i$-safe for I_{i-1}. By construction, all supernets of I_i terminate on the right boundary of I_i and shadownets terminate beside their respective supernets on the right boundary. By construction of I_{n+l-1}, the arrangements of the supernets in the last column of I_{n+l-1} is as given in Figure 11. V_h^l, V_i^l, and V_j^l are on a track in G_2, X_l and Y_l are in this order on tracks in G_4. We show that in any routing for I_{n+l}, Y_l changes its track to a track in G_2 within $D_{n+l,1}$, $D_{n+l,2}$, or $D_{n+l,3}$. Suppose that Y_l does not change its track in one of these channel portions. Then, Y_l is below X_l in the first column of $D_{n+l,4}$. And by Lemma 2.4, there is no such routing for I_{n+l}.

Now, suppose that Y_l changes its track with V_h^l within $D_{n+l,1}$. Then, V_h^l is on a track in G_4 and V_i^l and V_j^l are on tracks in G_2 in the first column of $D_{n+l,2}$. By construction, V_h^l is on a track in G_4 and V_i^l and V_j^l are on tracks in G_2 on the right boundary of I_{n+l} in any such routing. Furthermore all the following shuffle-checks are routable and Y_l is above X_l. Hence, there is a routing in this case.

Suppose that Y_l does not changes its track with V_h^l within $D_{n+l,1}$. Then V_h^l is on a track in G_2 on the right boundary of I_{n+l}, and either V_i^l or V_j^l is on a track in G_4. The proof that such routings exist is similar to the previous paragraph. ∎

To finish our construction for Theorem 3.1, we extend the instance $I_{n+m} = \{k, p, \mathcal{N}\}$ to the instance $I = \{k, q, \mathcal{N}\}$ in the following way. For each variable v_i check if all supernets V_i^l of clauses C_l with $v_i \in C_l$ are on tracks between H_i and V_i. We can do that by

FOR $i = 1$ TO n

FOR all l with $v_i \in C_l$

check $V_i^l < L_i$

check $H_i < V_i^l$

END

check $H_i < L_i$

END

Algorithm 1: The algorithm to extend $I_n + m$ to I. We check if all V_i^l of clauses C_l with variable v_i are routed between H_i and L_i.

successively applying the extension of Lemma 2.4 such that $V_i^l < L_i$ and $H_i < V_i^l$. The exact way to do that is shown in Alogrithm 1.

Claim 3.3. *A routing R' for I exists if and only if a routing R for I_{n+m} exists such that for all $i \in \{1, \ldots, n\}$ and each $l \in \{1, \ldots, m\}$ with $v_i \in C_l$, V_i^l is on a track between the tracks of H_i and L_i on the right boundary.*

Proof(Claim 3.3): We first show that there is a routing if the given condition is met. In this routing, for each $i \in \{1, \ldots, n\}$ there is some $h_i \in \{1, 3\}$ such that H_i is on a track in G_{h_i}, V_i^l is on a track in G_{h_i+1} for all l with $v_i \in C_l$, and L_i is on a track in G_{h_i+2}. This routing is such that in each of the checks $V_i^l < L_i$ and $H_i < V_i^l$ the supernets do not change their tracks. Hence there is a routing in any of these checks. There is also a routing for the last check, $H_i < L_i$.

Now we show that there is no routing if the condition is not met. Suppose that there is a routing R for I such that there exists some $l \in \{1, \ldots, m\}$ with $v_i \in C_l$ and V_i^l is not between H_i and L_i. We consider the first such supernet that is checked in the channel (the leftmost check. This supernet has minimal superscript l for a given i). Suppose that L_i, H_i are on tracks of G_1, G_3 at column $p^{\leftarrow}$. Then, since V_i^l is the first net in the checking of v_i that is not between G_1 and G_3, L_i and H_i are on tracks of $G_1 \cup G_2 \cup G_3$ and V_i^l is on a track in G_4. The following extension $V_i^l < L_i$ demands that L_i routed below V_i^l which leads to a contradiction. Hence, such a routing does not exist.

Suppose that L_i, H_i are on tracks of G_3, G_5 at column $p^{\leftarrow}$. Then, L_i, H_i are on tracks of $G_3 \cup G_4 \cup G_5$ before the check and V_i^l is on a track of G_2 (see Figure 12). If V_i^l does not change its track with L_i at the first check $V_i^l < L_i$, there is no routing for the second check

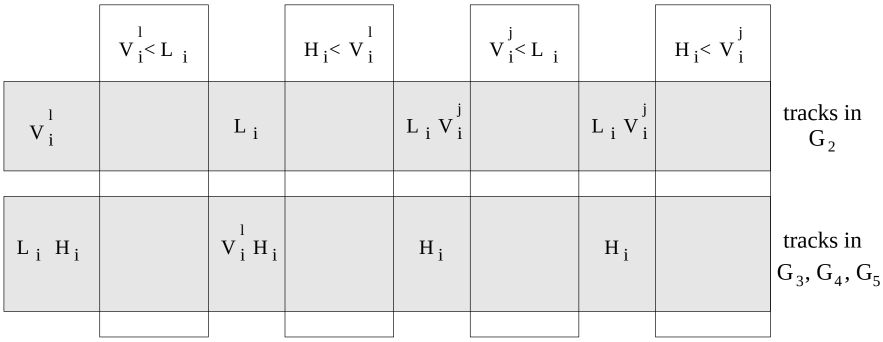

Figure 12: No routing exists if the truth-assignment of the global variable (done by L_i and H_i) does not match the truth assignment of the varibles in the clauses (V_i^l). A single check is shown for the case that L_i, H_i are on tracks of $G_3 \cup G_4 \cup G_5$ and V_i^l is on a track in G_2.

$H_i < V_i^l$. Hence, after the second check, L_i is on a track of G_2 and H_i is on a track of $G_3 \cup G_4 \cup G_5$. If C_l is the last clause that contains v_i, by Algorithm 1, a check $H_i < L_i$ follows which leads to a contradiction since such a routing does not exist. Suppose that there is another clause C_j with $v_i \in C_j$ that is checked right after the considered extension. V_i^j is routed on a track above L_i because of the check $V_i^j < L_i$. Hence, V_i^j is on a track in G_2 after this check, but the following $H_i < V_i^j$ demands that V_i^j is routed on a track below H_i which leads to a contradiction. Hence, such a routing does not exist. ∎

It remains to show that there exists a routing for I if and only if there is a $\mathcal{C}$-satisfying truth assignment for the variables in Σ such that there is exactly one true literal in each clause.

Suppose that such a truth-assignment exists. Consider the tracks of H_i, L_i on the right boundary of I_{n+m}. By Claim 3.1 and 3.21 we can find a routing for L_i, H_i such that H_i is on a track in G_1 and L_i is on a track in G_3 for all i with $v_i = false$ in the truth-assignment and furthermore H_i is on a track in G_3 and L_i is on a track in G_5 for all i with $v_i = true$ in the truth-assignment. Since in each clause exactly one of the variables is true, by Claim 3.2 we can find a routing such that for all $i \in \{1, \ldots, n\}$ and all l with $v_i \in C_l$ V_i^l is at a track in G_2 if $v_i = false$ and at a track in G_4 if $v_i = true$ in the truth-assignment. By Claim 3.3 there is a routing for I.

Suppose that there exists a routing R for $I = (k, p, \mathcal{N})$. The routing remains valid for the portion of I_{n+m}. Denote this routing of I_{n+m} with R_{m+n}. By Claim 3.1 and 3.21, H_i and L_i are either at tracks of $G_1 \cup G_3$ or at tracks of $G_3 \cup G_5$ on the right boundary of I_{n+m} in R_{m+n}. In the first case set $v_i = false$, in the latter set $v_i = true$. Consider this truth-assignment for varibles in Σ. By Claim 3.3, for all l with $v_i \in C_l$ the net V_i^l is between H_i and L_i on the right boundary of I_{n+m} in R_{m+n}. By Claim 3.2, for each l, exactly one supernet of V_h^l, V_i^l, V_j^l is on a track in G_4 and the other supernets are on tracks in G_2 on the right boundary of I_{n+m} in R_{m+n}. Hence, exactly one variable in each clause is true in this truth-assignment. ∎

Theorem 3.2. *The general knock-knee channel-routing problem with 3-terminal nets is NP-complete.*

Proof: It is easy to reduce KKRB with 3-terminal nets to the general knock-knee channel-routing problem . Let $I' = (k, p, \mathcal{N})$ be an instance of KKRB. We construct an instance I of the knock-knee channel-routing problem the following way: Replace the i-th net of the form $(a_1, \ldots, a_n, r)$ with the net $(a_1, \ldots, a_n, t_{p+i})$ (there are k such nets). Then, let instance I be the union of all the nets of the supernets in $\mathcal{N}$. Clearly, there is a routing for I if and only if there is a routing for I'. ∎

4 Conclusions

The main result of this paper is that Knock-knee channel-routing is NP-complete even if at most 3-terminal nets are involved. Polynomial time algorithm are known for Knock-knee channel routing with 2-terminal nets. Hence, this paper gives a rather sharp boundary of intractability for channel-routing in the knock-knee mode.

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